Curve Construction with Hermite Spline Interpolation

In numerical analysis, a cubic Hermite spline or cubic Hermite interpolator is a spline where each piece is a third-degree polynomial specified in Hermite form: that is, by its values and first derivatives at the end points of the corresponding domain interval. Hermite curves are very easy to calculate but also very powerful.

Bootstrap

From a bootstrap from instruments defined on the tenor $[0 = T_0, T_N]$, we get the average instantaneous forward rates on the intervals $[T_i, T_{i+1}]$

$$\bar{f}_i = \frac{\int_{T_i}^{T_{i+1}} f(u)du}{T_{i+1} - T_i}$$

Interpolation

We define the four Hermite functions as three order polynomials having the following properties:

$H_1(0) = 1 \quad H_1(1) = 0 \quad H_1'(0) = 0 \quad H_1'(0) = 0$

$H_2(0) = 0 \quad H_2(1) = 1 \quad H_2'(0) = 0 \quad H_2'(0) = 0$

$H_3(0) = 0 \quad H_3(1) = 0 \quad H_3'(0) = 1 \quad H_3'(0) = 0$

$H_4(0) = 0 \quad H_4(1) = 0 \quad H_4'(0) = 0 \quad H_4'(0) = 1$

We have:

$H_1(u) = 2u^3 - 3u^2 + 1$

$H_2(u) = -2u^3 + 3u^2$

$H_3(u) = u^3 - 2u^2 + u$

$H_4(u) = u^3 - u^2$

Let $I(t) = \int_0^t f(u)du$

We can express the value of I, as well as its first order derivative, on the tenor.

$$I(T_i) = \sum_{j=0}^{i-1} \bar{f}_j (T_{j+1} - T_j)$$

$$I'(T_i) = f(T_i) = \frac{\bar{f}_{i-1}(T_{i+1} - T_i) + \bar{f}_i (T_i - T_{i-1})}{T_{i+1} - T_{i-1}} \quad I'(T_N) = \bar{f}_{N-1}$$

Inside each interval $[T_i, T_{i+1}]$, we use the Hermite spline interpolation:

$$I(t) = I(T_i)H_1(s) + I(T_{i+1})H_2(s) + I'(T_i)(T_{i+1} - T_i)H_3(s) + I'(T_{i+1})(T_{i+1} - T_i)H_4(s)$$

$$s = \frac{t - T_i}{T_{i+1} - T_i}$$

A better interpolation

The way we choose $I'(T_i)$ is somewhat arbitrary.

$$I'(T_i) = f(T_i) = \frac{\bar{f}_{i-1}(T_{i+1} - T_i) + \bar{f}_i (T_i - T_{i-1})}{T_{i+1} - T_{i-1}} \quad I'(T_N) = \bar{f}_{N-1}$$

Curve properties

By construction, the instantaneous forward rates are continuous.

Reference

http://www.cubic.org/~submissive/sourcerer/hermite.htm